The Factorization of the Adjugate of a Finite Matrix*

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Let A be an $n \times n$ matrix with coefficients in an algebraically closed field. Its adjugate A^{adj} has as determinant the $(n-1)$st power of the determinant of A. This elementary fact is well known. It does not seem to have been pointed out that the matrix A^{adj} can be factorized into the product of $n-1$ matrices each with $\det A$ as its determinant. This can be achieved in various ways. Here a factorization will be given in the form

$$A^{\mathrm{adj}} = \prod_{i=2}^{n} A_i, \qquad \det A_i = \det A,$$

so that in the resulting factorization of

$$\det A \cdot I = A \cdot A^{\mathrm{adj}} = \prod_{i=1}^{n} A_i$$

with $A = A_1$, every factor A_i plays the same role.

Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be the characteristic polynomial of A and let $\alpha_1, \ldots, \alpha_n$ be the characteristic roots. Let $\lambda_2, \ldots, \lambda_n$ be the zeros of $x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}$ and $\lambda_1 = 0$.

THEOREM 1. *The adjugate of A is the product $\prod_{i=2}^{n} (\lambda_i I - A)$ and* $\det(A - \lambda_i I) = \det A$.

Proof. We first assume $\det A \neq 0$. Then

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$$A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I$$
$$= - a_n A^{-1} = (-1)^{n-1} \det A \cdot A^{-1} = (-1)^{n-1} A^{\text{adj}}.$$

This proves the first part of the theorem. In order to prove the second part consider the algebraic equation of degree n in λ:

$$\prod_{i=1}^{n} (\alpha_i - \lambda) = \prod_{i=1}^{n} \alpha_i. \tag{1}$$

This equation has n roots λ. They are $\lambda_1 = 0$ and the quantities $\lambda_2, \ldots, \lambda_n$ defined above. On the other hand, $\prod \alpha_i = \det A$ and $\prod (\alpha_i - \lambda) = \det(A - \lambda I)$. This proves the second part.

The case when A is singular follows by specialization. No inverses occur in the factorization.

THEOREM 2. *Denote $A - \lambda_i I$ by A_i, $i = 1, \ldots, n$. The factorization process of Theorem* 1 *applied to A_i, $i > 1$, yields the A_j, $j \neq i$.*

Proof. Consider, e.g., $i = 2$. Then find λ such that

$$\prod_{i=1}^{n} (\alpha_i - \lambda_2 - \lambda) = \prod_{i=1}^{n} (\alpha_i - \lambda_2).$$

Equation (1) implies $\prod_{i=1}^{n} (\alpha_i - \lambda_2 - \lambda) = \prod_{i=1}^{n} \alpha_i$. Hence the λ's obtained here are the λ's of Theorem 1 shifted by λ_2.

Remarks. 1. In the matrix whose (i, k) element is $\alpha_i - \lambda_k$ each row or column has the same product. For the columns this follows from (1). For the rows it follows by an easy computation.

2. Special examples are:

(1) $x^n + a$. Here all the λ's are zero.

(2) $(x - c)^n$. This is the characteristic polynomial of a Jordan block of dimension n with characteristic root c. Here the λ's are the products of c and $1 - \zeta_n$ where ζ_n runs through the zeros of $x^n - 1$.

(3) Let the field have a characteristic which does not divide $n - 1$. Let A be a Jordan block with characteristic root c; then A^{adj} is the same Jordan block with characteristic root c^{n-1}. Such a block can be expressed as the $(n - 1)$st power of a triangular matrix

$$\begin{pmatrix} c & x_2 & x_3 & \cdots & x_n \\ 0 & c & x_2 & \cdots & x_{n-1} \\ 0 & 0 & c & \cdots & x_{n-2} \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}$$

where the x_i can be determined recursively (see [2]). This was generalized by R. C. Thompson (private communication) to arbitrary matrices in Jordan normal form.

3. For factorizations of det A, see also the recent paper of Jurkat and Ryser [1].

REFERENCES

1 W. B. Jurkat and H. J. Ryser, Matrix factorizations of determinants and permanents, *J. Algebra* **3**(1966), 1–27.
2 O. Taussky, The factorization of the adjugate matrix, *Notices Amer. Math. Soc.* **13**(1966), 492.

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